Turbulence

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Abstract

As part of the proseminar in theoretical physics on out of equilibrium physics, I chose to speak about turbulence. Due to the problem's difficulty, approaches are mostly of statistical nature and results from first principle remain elusive. The treatment here is separated into fully developed turbulence, where Kolmogorov theory is most established, and the transition to turbulence, where a modeling approach to find universality is studied. It is the goal throughout to find hints of universality. In Kolmogorov's theory for developed turbulence, I derive the famous $p/3$ and $-5/3$ scaling laws from Kolmogorov's hypotheses. I discuss its limits from experimental disagreements and explore a possible adjustment of the scaling laws. Consideration of complete and incomplete similarity arguments leads to an anomalous scaling and the scale interference of large length scales, thereby complicating the search for universality. A possible reconciliation is motivated via the comparison to critical phenomena, where a similar problem of scale interference was solved by the renormalization group approach. For the transition to turbulence, I discuss experimental results and observations of puff dynamics. Puff decay and puff splitting resemble the nature of population decay and splitting in predator-prey dynamics. Ultimately, I motivate the identification of the phenomenon with the universality class of directed percolation, a universality class of out of equilibrium phase transitions. Hence, a sense of universality and simple scaling is established for the onset of turbulence.

Contents

1 Introduction

Throughout this treatment we will be concerned with the behaviour of fluids, categorized in the field of fluid dynamics. The governing equation of fluids is the *Navier-Stokes equation*. It is an approximation of Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, following a promotion of a particle description $\boldsymbol{u}(t)$ to a field description $\boldsymbol{u}(\boldsymbol{x}, t)$ by letting the number of particles in a volume of interest go to infinity. A general force $\boldsymbol{F}$ in such a field is typically dissected into a pressure gradient, $\boldsymbol{\nabla} p$, a viscous stress, $\mu\nabla^2\boldsymbol{u}$ (μ: dynamic viscosity) and body forces, $\rho\boldsymbol{f}$. Using the Lagrange derivative, $\frac{D\boldsymbol{u}}{Dt} = \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, the Navier-Stokes equation is written as

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right) = -\boldsymbol{\nabla} p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f}, \tag{1}$$

where note that mass m was promoted to a notion of density ρ within the field description. Roughly, a fluid's flow is classified as being of laminar or turbulent nature. Laminar flow is characterized as being smooth and non-mixing with a parallel stream line profile. Laminar flow is a steady and stationary state. Typically, approximate laminar flow is observed from the water tap for low stream intensities, in mean flows of river channels or in pipes at low flow velocities. For increasing flow velocities, Reynolds famously observed fluid flows to become more and more chaotic (Reynold's pipe flow experiment 1883). This chaotic flow of a fluid is what is generally called *turbulence*. It is generally the state of fluids much more abundantly observable in nature than laminar flow, e.g. in the swirly movement of ocean dynamics or in the diffusive motion of a stirred cup of coffee. Turbulent flow is a highly out-of-equilibrium phenomenon, changing vastly with time and consisting of intermittent fluctuations.

Turbulence is of great interest in many fields of research: in mechanical engineering, where it is the reason for amplifying drag in aerodynamics - in medicine, where turbulent blood flow is thought to lead to aneurysm - in earth sciences, for weather and climate predictions - in pure mathematics, where the search of general closed-form solutions to the NS equations is part of a millenium problem - and of course in physics, where its abundance in nature aches for a universal description. However, the phenomenon of turbulence has long proven to be resistant to quantitative studies. Famous quotes from great physicists (Feynman, Heisenberg and more) promote the idea of it being the last unsolved problem of classical physics. To which extent it is unsolved or what it would mean to solve turbulence is a philosophical debate of its own. Roughly speaking, turbulence misses a theory from first principles built on a general, closed-form solution of the NS equation. Such a solution is difficult to find due to the non-linear (highly interacting) nature of the equation. Hence, to this date theories are based on statistical approaches and often lack full conclusive strength.

I will present the standard approach in section 2, which encapsulates the most established theory related to turbulence, called *Kolmogorov Theory*, which is a description of fully developed turbulence. In section 2.4, I will illustrate a connection of turbulence to the field of critical phenomena. Based on that, I will introduce approaches to the onset of turbulence by means of out-of-equilibrium statistical physics in section 3.

2 Fully Developed Turbulence

This section is largely based on the book '*Turbulence and the Legacy of A.N. Kolmogorov (1995)*' by Uriel Frisch [1].

2.1 Reynolds Number

As mentioned in the general introduction above, the governing equation of turbulent flow (and fluid flows in general) is the Navier-Stokes equation 1. An important characterizing parameter of the NS equation and one encountered throughout the discussion around turbulence is the so-called *Reynolds number* Re. Qualitatively it is a ratio of inertial forces to viscous forces. The Reynolds number is a non-dimensional quantity defined by non-dimensionalizing the NS equation. Each term in the NS equation has units of a body force, i.e. force per unit volume. To reach a non-dimensional form, we multiply the equation with a term of inverse units containing characteristic system scales, e.g. $\frac{L}{\rho V^2}$, where L is a characteristic length sale and V is a characteristic velocity. Defining non-dimensional quantities as $u_0 = \frac{u}{V}$, $p_0 = \frac{p}{\rho V^2}$, $f_0 = f\frac{L}{V^2}$, $\frac{\partial}{\partial t_0} = \frac{L}{V}\frac{\partial}{\partial t}$ and $\nabla_0 = L\nabla$, we can rewrite the NS equation as

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f} \quad \xrightarrow{non-dim} \quad \frac{D\boldsymbol{u}_0}{Dt_0} = -\boldsymbol{\nabla}_0 p_0 + \frac{1}{\mathrm{Re}}\nabla_0^2\boldsymbol{u}_0 + \boldsymbol{f}_0,$$

where $\mathrm{Re} := \frac{LV}{\nu}$ and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.
The battle of inertial to viscous forces is a key player in the differentiation between laminar and turbulent flow. In laminar flow, viscous stress between different layers of the fluid dampen fluctuations that dance out of line any present mean flow resulting in a parallel stream profile. As the fluid's velocity, V, (or the system's length scale L) increases, inertial effects start to dominate viscous effects and fluctuations spread without considerable dampening resulting in a turbulent flow field. Quantitatively, this battle can be related to the fact that as $\mathrm{Re} \to \infty$ the viscous term in the NS equation becomes negligible and the non-linear, interacting nature of the equations gain importance.

2.2 Energy Cascade

In 1922, Lewis Richardson published a book[2], where he wrote about his efforts towards weather prediction. In it you will find qualitative analysis of the behaviour of a fully turbulent fluid. He noticed that turbulent regions tend to dissipate back into a laminar state if not continually injected with external energy (s.a. a hand whirling in a pool of water). He then imagined an idealized steady state in a bounded volume, where fully turbulent flow would be sustained by a mean external energy injection $\bar{\epsilon}_{\mathrm{in}}$ acting against and compensating the energy dissipation rate $\bar{\epsilon}_{\mathrm{diss}} = \bar{\epsilon}_{\mathrm{in}} = \bar{\epsilon}$. In this setup, Richardson then formulated the fundamental idea of an energy cascade: The injected energy turns over into eddies comparable to the system's scale (called *integral scale*, e.g. the diameter of a coffee cup, or the depth of the sea), which carry most of the injected kinetic energy. Somewhere down the line this kinetic energy will be converted into thermal energy, when the eddy sizes are comparable to the molecular scale (called *Kolmogorov scale*). Richardson then postulated that intermediate length scales (called *inertial scale*) feel no direct forcing of the large scale, and do not interact significantly with the molecular scale. There, eddies interact only with eddies of similar size, leading to a non-dissipative cascade of larger eddies to smaller eddies. Richardson [2] summarized the qualitative analysis in a poem (adapted from '*Siphonaptera*' by Augustus de Morgan):

'Big whirls have little whirls
that feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity.'

It is this idea of a non-dissipative, cascading intermediate (inertial) range that is most out there and serves as the intuitive guess we see at the start of many theories in physics.

2.3 Kolmogorov Theory

In 1941, it was Kolmogorov [3] who quantified the qualitative energy cascade observation of Richardson by formulating three hypotheses in the large Reynolds number limit (i.e. the inertia dominated, turbulent regime). I will restate the hypotheses and harvest their qualitative and quantitative consequences step by step.

Hypothesis 1: On sufficiently small scales, the velocity field of turbulence is isotropic.

In other words, eventhough the large scale fluctuations usually are strongly anisotropic (due to different boundary influences and mean flows), Kolmogorov presumed that the smallest fluctuations of the flow anywhere in the flow field look the same (homogeneous) and that furthermore a ficticious observer at these small scales could not discern any direction from another when turning their head (isotropy). Note that this loss of larges scale geometrical information quantifies a first notion of universality, meaning that perhaps turbulent flows in different geometrical setups could show similar behaviour.

Hypothesis 2: On small scales and high Re, statistical properties are uniquely and universally determined by the length scale l, the mean dissipation rate (per unit mass) $\bar{\epsilon}$ and the viscosity ν.

Roughly, small scales means that the respective length scales l should be much smaller that the size of the geometry Λ, i.e. $l << \Lambda$. We quantify the notion of sufficiently small scales further by constructing a length scale η_K solely from $\bar{\epsilon}$ and ν via dimensional considerations, i.e. $\eta_K = F(\bar{\epsilon}, \nu)$ (for F some function). The dimensions of the energy dissipation rate and kinematic viscosity are $[\epsilon] = V^2/T = L^2/T^3$ and $[\nu] = L^2/T$, respectively. Then, the so-called *Kolmogorov length scale* is defined by combining powers of $\bar{\epsilon}$ and ν, such that the multiplication results in units of length:

$$\eta_K \sim \left(\frac{\nu^3}{\bar{\epsilon}}\right)^{1/4}. \tag{2}$$

Note, that for decreasing viscosity or increasing energy injection rate (i.e. increasing Re) these small scales become smaller and smaller compared to the fixed, large scale geometry. Hence, for increasing Reynolds numbers the intermediate, inertial scales spread over a larger and larger range.

Similarly, we can construct a characteristic quantitity for the velocity fluctuations u_K on small length scales (i.e. small eddies), also called the *Kolmogorov velocity*. From the hypothesis, we make the Ansatz that the fluctuations are a function of only $\bar{\epsilon}$ and ν, i.e. $u_K = F(\bar{\epsilon}, \nu)$. From dimensional analysis, we arrive at the expression

$$u \sim (\bar{\epsilon}\nu)^{1/4}. \tag{3}$$

The corresponding, characteristic Reynolds number for this small scale regime of length scale η_K and small velocity fluctuations u (i.e. small eddies),

$$\mathrm{Re}_K = \frac{\eta_K u_K}{\nu} \sim 1,$$

where we insert the expression for η_K and u_K. By construction, this indicates that at these small scales viscous effects are no longer negligible compared to the inertial effects, thus encapsulating the idea of dissipation at the Kolmogorov scale.

Hypothesis 3: On small scales and infinite Re, all statistical properties are uniquely and universally determined by the length scale l and the mean dissipation rate (per unit mass) $\bar{\epsilon}$.

Note the subtle difference of infinite Re instead of 'large' Re. This implicit limit will concern us further down the road. For now, it merely imposes an additional condition on the range of permitted length scales l. Namely, the limit $\mathrm{Re} \to \infty$ leads to an ever decreasing Kolmogorov length η_K, where energy is dissipated. Since the geometry size Λ is fixed, the respective length scales l implied by the hypothesis lay in the range $\eta_K << l << \Lambda$.
Generally, statistical properties (as mentionned in hypotheses 2 & 3) are summarized special functions of a systems fluctuations. These functions are called structure functions S_p of order p (also called moments)

$$S_p(l) = \langle \Delta u(l)^p \rangle, \tag{4}$$

where $\Delta u(l) = u(r+l) - u(l)$ are fluctuations in the velocity field on a length scale l and $\langle\rangle$ is an appropriate average (ergodic, temporal, spatial).

Translating the hypothesis, we make an Ansatz for S_p only to depend on the respective length scale l and the mean energy rate $\bar{\epsilon}$

$$S_p \sim \bar{\epsilon}^{\alpha} l^{\beta}.$$

The dimensions of the structure function are $[S_p] = [(\Delta u)^p] = V^p = \left(\frac{L}{T}\right)^p$. By dimensional analysis, we can determine the exponents α and β and arrive at a scaling relation for S_p,

$$S_p \sim \bar{\epsilon}^{p/3} l^{p/3}.$$

This relation is what is often called Kolmogorov's *K41* law.
We can rewrite this relation in order to arrive at one of the most famous results of turbulence. We can rewrite fluctuations on a length scale l by its Fourier pendant, i.e. fluctuations of corresponding wavelengths $k = 1/l$. The K41 law can then be rewritten as

$$S_p \sim \epsilon^{p/3} k^{-p/3}.$$

Note, that the structure function of 2nd order, S_2, is just the mean kinetic energy E in fluctuations up to wavelengths k (see Eq. 4). We define the spectrum $E(k)$ of kinetic energy per wavelength k as

$$E = \int E(k)\, dk.$$

Using the K41 law, we can write a relation for the energy spectrum by accounting for another dimension of k in the dimensional analysis:

$$E(k) \sim \frac{1}{k}\, S_2 \sim \bar{\epsilon}^{2/3}\, k^{-5/3}$$

This result is called *Kolmogorov's* 5/3*-law.*

However, numerical simulations of the NS equations show deviations from the K41 scaling law (see fig. 1).

We see that the prediction for exponent for the 2nd order structure function is very close to the values from simulations. Hence, the 5/3 law is very close to the truth, which is why it has been proven so valuable in a variety of engineering applications and has gained the mentioned fame status. The fact that the exponents differ from Kolmogorov's predictions for high order structure functions hints at some ignorance in the formulation of Kolmogorov's hypotheses. To gain quantitative insight, a quick excursion to introduce the concept of similarity is needed.

Interlude: Similarity[1].

Any physically significant relationship can be written as a quantity a being a function of governing and non-governing parameters, a_i and b_i,

$$a = f(a_1, ..., a_k, ..., b_1, ..., b_m). \tag{5}$$

A result from dimensional analysis called the Π-theorem states that any such physical law can be rewritten in non-dimensional form

$$\Pi = \Phi\left(\Pi_1, ..., \Pi_m\right), \tag{6}$$

[1] This interlude is largely based on the book '*Scaling*' by G. Barrenblatt [5]

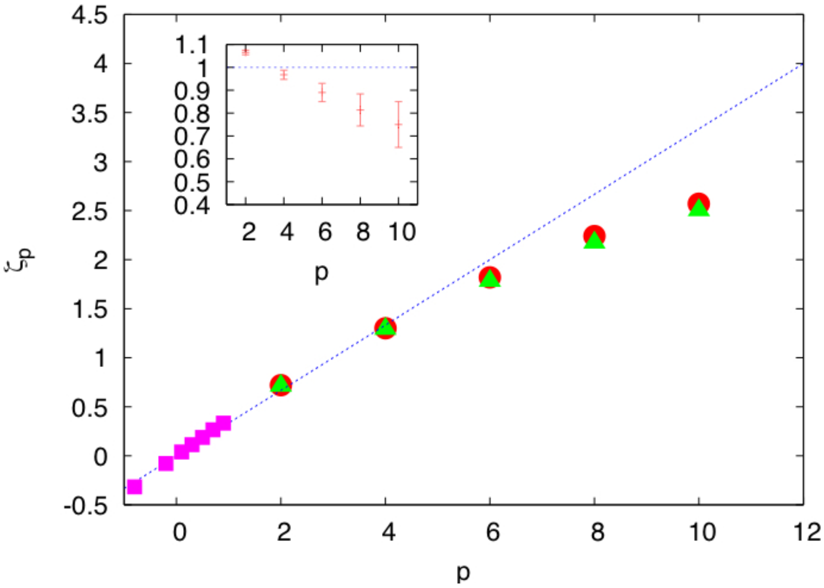

Figure 1: Value of the exponent ζ_p in the scaling law $S_p \sim (\bar{\epsilon} l)^{\zeta_p}$. The dotted line corresponds to the K41 prediction that $\zeta_p = p/3$. The purple squares, red circles and green triangles represent values measured from direct numerical simulations of the NS equation. The deviation from the K41 value is what is often called *anomalous scaling*. (R. Benzi, U. Frisch (2010) [4]

where $\Pi = \frac{a}{a_1^p \cdots a_k^r}$ and $\Pi_i = \frac{b_i}{a_1^{p_i} \cdots a_k^{r_i}}$. Intuitively, the theorem quantifies that physically significant laws should not depend on a choice of units. Combining eq. 5 and 6, we can write

$$f(a_1, a_2, ..., b_1, b_2, ...) = a_1^p \cdots a_k^r \Phi(\frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, ..., \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}), \tag{7}$$

which is sometimes called the condition of generalized homogeneity.

Definition (Complete Similarity)
Taking the limit of $\Pi_{t+1}, ..., \Pi_m \to 0$ or ∞, we say a law obeys *complete similarity* iff the function Φ is finite-valued in the respective limit, i.e.

$$\lim_{\Pi_{t+1},...,\Pi_m \to 0 \text{ or } \infty} \Phi(\Pi_1, ..., \Pi_m) = C\,\tilde{\Phi}(\Pi_1, ..., \Pi_t),$$

where C is a constant.

Definition (Incomplete Similarity)
Taking the limit of $\Pi_{t+1}, ..., \Pi_m \to 0$ or ∞, we say a law obeys *incomplete similarity* iff the function Φ is vanishing or singular in the respective limit and then we to write the dependencies as power laws

$$\lim_{\Pi_{t+1},...,\Pi_m \to 0 \text{ or } \infty} \Phi(\Pi_1, ..., \Pi_m) = \Pi_{t+1}^{\alpha_{t+1}} \cdots \Pi_m^{a_m}\,\tilde{\Phi}(\Pi_1, ..., \Pi_t).$$

Qualitatively, a law is completely similar under $\Pi_{t+1}, ..., \Pi_m$ if we can neglect the parameters $\Pi_{t+1}, ..., \Pi_m$ for big or small enough values. Conversely, it is incompletely similar if the law still depends on the parameters $\Pi_{t+1}, ..., \Pi_m$ even in their respective asymptotics. The notions of complete and incomplete similarity are also tightly connected and often referred to interchangeably as *scale invariance* and *scale interference*, respectively.

Kolmogorov Revisited

We take another look at Kolmogorov's approach to the energy cascade, while keeping an eye for implicit assumption of similarity. In general (before using any of Kolmogorov's hypotheses), starting out at the largest, integral scale we include all possible dependencies for the statistic properties. That includes the respective length scale of a particular eddy, l, the viscosity ν, the size of the geometry Λ and the energy injection/dissipation rate $\bar{\epsilon}$. In other words, S_p is a function of l, ν, Λ and $\bar{\epsilon}$:[2]

$$S_p = f(l,\, \eta_K,\, \Lambda,\, \bar{\epsilon}), \tag{8}$$

where we capture the ν dependency in the Kolmogorov length scale $\eta_K = \left(\frac{\nu^3}{\bar{\epsilon}}\right)^{\frac{1}{4}}$. We can identify the above expression with the one for a general physical law in the previous interlude, eq. 5. Invoking the Π-theorem, 6, we write the above in the form of generalized homogeneity, 7,

$$S_p = \bar{\epsilon}^{p/3} l^{p/3}\, \Phi\left(\frac{l}{\eta_K},\, \frac{l}{\Lambda}\right).$$

Kolmogorov's second and third hypotheses include two limits. The second hypothesis is a statement in the regime of small scales, $l << \Lambda$. This is restated as a limit $\frac{l}{\Lambda} \to 0$. Furthermore, the second hypothesis states that in this limit, S_p only depends on l, ν and $\bar{\epsilon}$. Looking at the above equation, we translate this statement into

$$\lim_{l/\Lambda \to 0} = \bar{\epsilon}^{p/3} l^{p/3}\, C_1\, \Phi\left(\frac{l}{\eta_K}\right),$$

i.e. that the limit is finite valued and hence it is implicitly assumed that S_p obeys complete similarity under the parameter $\frac{l}{\Lambda}$.

The third hypothesis is a statement in the regime of small scales *and* infinite Re, i.e. Re $\to \infty$, which is equivalent to the limit $\frac{l}{\eta_K} \to \infty$. Furthermore, the third hypothesis states that in this limit, S_p only depends on l and $\bar{\epsilon}$. We again translate this statement into the above equation, i.e.

$$\lim_{l/\eta_K \to \infty,\; l/\Lambda \to 0} S_p = \bar{\epsilon}^{p/3} l^{p/3}\, C_1 C_2.$$

Note that $S_p \sim \bar{\epsilon}^{p/3} l^{p/3}$ is equivalent to saying $S_p = C\, \bar{\epsilon}^{p/3} l^{p/3}$ for some constant C (in this case $C = C_1 C_2$). Again, the above hypothesis assumes a finite-valued limit of S_p for

[2] Of course, these dependencies are still highly debatable and already represent a form of assumption and approximation. But to illustrate a limit of Kolmogorov's assumptions, this Ansatz suffices.

$\frac{l}{\eta_K} \to \infty$ hence that S_p is completely similar under the parameter $\frac{l}{\eta_K}$.
We therefore realize that the Kolmogorov hypotheses implicitly imply complete similarity in both parameters.

As simulations show a deviation from the K41 prediction, it remains to identify which of the limits fails to behave under complete similarity. Qualitatively, the first limit can be read as a notion of scale invariance on small scales, meaning the flow field on small scales does not depend on the large scale geometry Λ. The second limit reads as the fluid (or spec. its statistical properties) becoming essentially inviscid for infinite Re. From the introductory discussion around the Reynolds number, we know that based on the NS equation the approximation of a fluid to be inviscid in the limit of infinite Reynolds numbers is reasonable. This fact is also well supported by experimental and numerical results. Commonly, it is the first limit of length scale invariance that is accused of being incompletely similar. Following the interlude on similarity, we need to account for this incomplete similarity by writing the first limit as

$$\lim_{l/\Lambda \to 0} S_p = \bar{\epsilon}^{p/3} l^{p/3} \left(\frac{l}{\Lambda}\right)^{\theta} \Phi\left(\frac{l}{\eta_K}\right).$$

Then invoking the completely similar second limit, we get

$$\lim_{l/\Lambda \to 0,\, l/\eta_K \to \infty} S_p = \bar{\epsilon}^{\frac{p}{3}} l^{\frac{p}{3}+\theta} \Lambda^{-\theta} C_2,$$

where θ is called the anomalous scaling. This anomalous exponent can be compared to the offset from the K41 prediction in fig. 1.
To conclude, we see that the system retains a sense of 'memory' of the large scale geometry Λ even on its small scales. This seemingly destroys our efforts to reach universality, meaning results independent of specific system geometry. There are a variety of approaches and active research to this date to include the anomalous scaling in the theory and restore universality. Rigorous and suitable techniques to derive such results from first principles are still to be found.

2.4 A Similar Story: Critical Phenomena

This section is largely based on the lecture notes by Prof. Mehran Kardar (MIT) on '*Statistical Mechanics II*' [6].

In this section, I will outline the history of studies related to critical phenomena and show how a lot of its developments resemble steps we've found in the above discussion around Kolmogorov's theory of turbulence.

The field of critical phenomena is concerned with the study of phase transitions. Well-known examples of phase transitions are the liquid-gas transition or the paramagnetic-ferromagnetic transition. Such phase transitions are characterized by an order parameter, which is a physical observable that is usually vanishing in one phase and non-zero in the other phase (e.g. the magnetization m for the para-ferro transition). Furthermore, the transition is characterized by a control parameter by which we move between two phases (e.g. the temperature T for the para-ferro transition). We call the point, where the order parameter changes from zero to a non-zero value the *critical point* (e.g. the critical temperature T_C for the para-ferro transition).

Assigned with the task of developing an according theory one would definitely be overwhelmed, as we would have to cope with the quantum complexity of a huge amount of interacting microscopic particles. However, the mere fact that we observe 'phases', i.e. collective ordered states of these particles, hints at the fact that perhaps sometimes a microscopic description is not necessary. Observational experience shows that these microscopic particles start to move collectively as a phase transition is approached (e.g. in the effect of *critical opalescence*). This collective behaviour indicates the presence of long wavelength fluctuations rendering the microscopic details unnecessary. Instead of trying to trace every detail and degree of freedom during a phase transition, Landau and Ginzburg proposed to focus on this long-ranged behaviour of particles.

Landau-Ginzburg Theory

Based on the above motivation, Landau and Ginzburg simplified the microscopic problem by replacing the magnetization m stemming from each particle by a coarse-grained field $\phi(x)$. In other words, the field $\phi(x)$ only allows for slowly varying, long wavelength fluctuations. Together with symmetry arguments (locality, rotational symmetry and translational symmetry), they proposed an effective form for the system's free energy

$$\beta F = \frac{t}{2}\phi^2 + u\phi^4 + ... + \frac{K}{2}(\nabla\phi)^2 + ... - h \cdot \phi.$$

From the free energy, a series of scaling laws for the thermodynamic properties near the critical point can be derived. For example, the magnetization in the para-ferro transition is predicted to scale as $m \sim |t|^{1/2}$ with temperature ($t = 1 - \frac{T}{T_C}$ is the reduced temperature). However, numerical simulations (and exact solutions like the Onsager solution for the Ising model in 2D) do not agree with the scaling exponent of value $\frac{1}{2}$.

Similarly, Ginzburg-Landau theory gives a prediction for the scaling of the correlation function

$$\langle\phi(x)\phi(y)\rangle \sim |x-y|^{-(d-2)},$$

which also disagrees with numerical simulations and shows an anomalous exponent θ. The origin of this deviance can be formulated in the scheme of similarity, like we did for Kolmogorov's theory of turbulence.
The units of the correlation function are $L^{-(d-2)}$. The condition of generalized homogeneity for this case is

$$\langle\phi(x)\phi(y)\rangle = |x-y|^{-(d-2)}\Phi\left(\frac{|x-y|}{a}\right),$$

where the initial dependence on the microscopic scales a is included. The act of coarse graining can be rewritten into the limit $\frac{|x-y|}{a} \to \infty$. Ginzburg-Landau theory then implicitly assumes a finite-valued limit of Φ in this limit. s.t.

$$\lim_{\frac{|x-y|}{a} \to \infty} \langle \phi(x)\phi(y) \rangle = C\,|x-y|^{-(d-2)}.$$

The experimental denial and introduction of an anomalous exponent hints at the fact that the function behaves incompletely similar, s.t.

$$\lim_{\frac{|x-y|}{a} \to \infty} \langle \phi(x)\phi(y) \rangle = |x-y|^{-(d-2)} \left(\frac{|x-y|}{a} \right)^{\theta} = |x-y|^{-(d-2)+\theta} a^{-\theta},$$

and a memory on the microscopic length scales is retained even on large scale fluctuations.

The reconciliation and explanation of universality in the case of anomalous scaling came eventually with the **renormalization group** approach introduced by Kenneth Wilson in 1982. It is an approach that embraces the fact that in certain problems (like critical phenomena) fluctuations persist out to macroscopic wavelengths, where furthermore fluctuations on all intermediate length scales remain important too. The strategy of the approach is to tackle each length scale, step by step. For critical phenomena, this means carrying out statistical averages over thermal fluctuations on the atomic scale and then moving to successively larger scales until fluctuations on all scales have been averaged out[7].

From the discussion of Kolmogorov's theory, we realize that turbulence is a problem of similar nature, where the macroscopic length scale and all intermediate length scales still remain important on small scales. Finding a renormalization group-like approach to sequentially tackle fluctuations on all length scales in turbulence is part of ongoing research, but conclusive results have yet to be found.

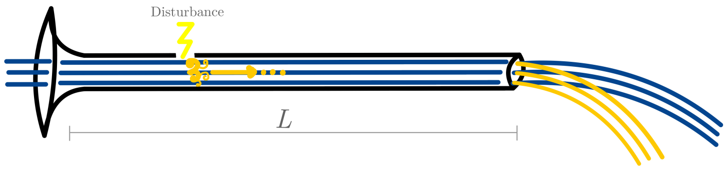

Figure 2: Experimental setup of Hof's experiment [9]. It consists of a pipe connected to a fluid reservoir (to the left) of length L with a controlled access to introduce a disturbance to the fluid flow. A binary measurement of the death or survival of a puff is done by measuring the fluid's outflow velocity at the pipe's outlet.

3 Transition to Turbulence

This section is largely based on the paper '*Turbulence as a Problem in Non-Equilibrium Statistical Mechanics*' by N. Goldenfeld et al. [8].

The systematic study of the transition to turbulence started with the famous pipe experiment by Reynolds in 1883. As he increased the fluid's flow velocity through the pipe, he observed a higher and higher occurance of 'flashes' of turbulent regions, now called *puffs*. This transition would run smoothly with increasing flow velocity until a fully turbulent flow region occupied the pipe. We could try to connect the turbulent transition to the notion of a critical phenomenon. We could identify the order parameter of the system by the turbulent fraction (i.e. the amount of turbulent compared to laminar regions) and further use the Reynolds number as the control parameter. A question that arises and to which I show a potential answer in the next section, is where to define the critical value of the Reynolds number in such a critical phenomenon.

3.1 Lifetime & Splitting of Puffs

Notable progress in the study of the transition to turbulence came from Hof's experiment [9] in 2008. He continued the study of the emergence of the above mentionned puffs in a sophisticated version of Reynold's pipe experiment. The setup (see fig. 2) is a pipe of length L connected to a reservoir, where the fluid's supply and velocity is controlled. Furthermore, there is a point of access at the pipe to introduce a controlled disturbance, i.e. a puff.

By measuring the fluid's outflow velocity at the pipe's ending, Hof was able to perform a binary measurement of wether the introduced puff survived until the end or died on the way. By doing lots of repetitions of such an experiment while also varying the flow velocity (Reynolds number Re) and the pipe length L, Hof arrived at a probability distribution $P(\mathrm{Re}, L)$ (see fig. 3). The graph reads as the probability of survival P for a given Reynolds number Re and length of pipe L (labeled t in the graph).

The e-fold lifetime $\tau(\mathrm{Re})$ of this probability function $P(\mathrm{Re}, t) = e^{\frac{t}{\tau(\mathrm{Re})}}$ is plotted in the graph on the right in fig. 3. Fitting the graph, it turns out the lifetime behaves as a

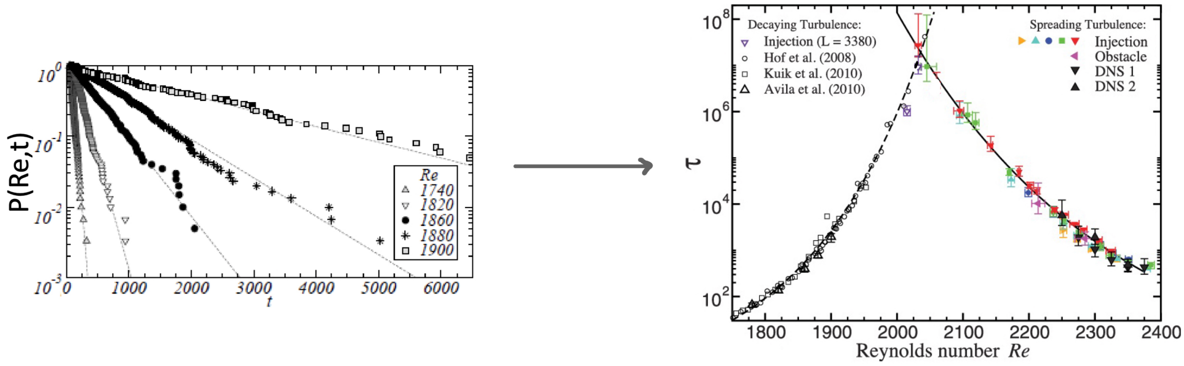

Figure 3: (left) The probability of survival for a variety of Reynolds numbers (portrayed by different shapes) and length of the pipe (here translated into a notion of time t). The e-fold lifetime of this distribution can be plotted as the lifetime of a puff (right, decaying turbulence). Similarly, measurements of splitting puffs lead to the graph on the mean time between splitting events of puffs. (Hof et al. 2011 [10])

superexponential function of Re, $\tau \sim e^{e^{\mathrm{Re}}}$. At this point, it almost seems hopeless to search for a notion of critical Re, since it seems like a fully turbulent regime would never stabilize. With increasing Reynolds number puffs will live (super)exponentially longer. However, if one waits long enough it would always decay. The answer to this problem comes from a second observation Hof made, which is the splitting of puffs. He found, that at high enough Reynolds numbers single puffs started to infect laminar regions close to them leading to new, 'baby'-puffs evolving independently from the 'mother' puff. This phenomenon is called *puff splitting*. Similar to before, Hof measured the second graph portrayed on the right of fig. 3 corresponding to the mean time between splitting events of puffs. He arrived at the result that the time between splitting events decreased superexponentialy for increasing Reynolds number. Now, we are presented with a possibility of defining a critical Reynolds number, namely the intersecting point between the lifetime of puffs and the mean time between splitting events of puffs. The justification for this choice is that to the left of this point it is more likely for a puff to decay than for it to split, while to the right of the point it is more likely for a puff to split than for it to decay leading to an ever growing region of turbulence.

3.2 Predator-Prey Dynamics and Directed Percolation

N. Goldenfeld [11] realized that a second phenomenon he was working on, predator-prey population dynamics of biophysics, showed similar behaviour to the effects discussed above. Isolated populations of predator and prey arise and collapse after some time. They tend to survive longer and longer for increasing birth rates, pendant to the lifetime of puffs for increasing Reynolds numbers. Furthermore, for high enough birth rates such populations start splitting into seperate, independent populations. The modeling of this behaviour leads to the same superexponential scaling found for puffs in turbulence (see fig. **??**).

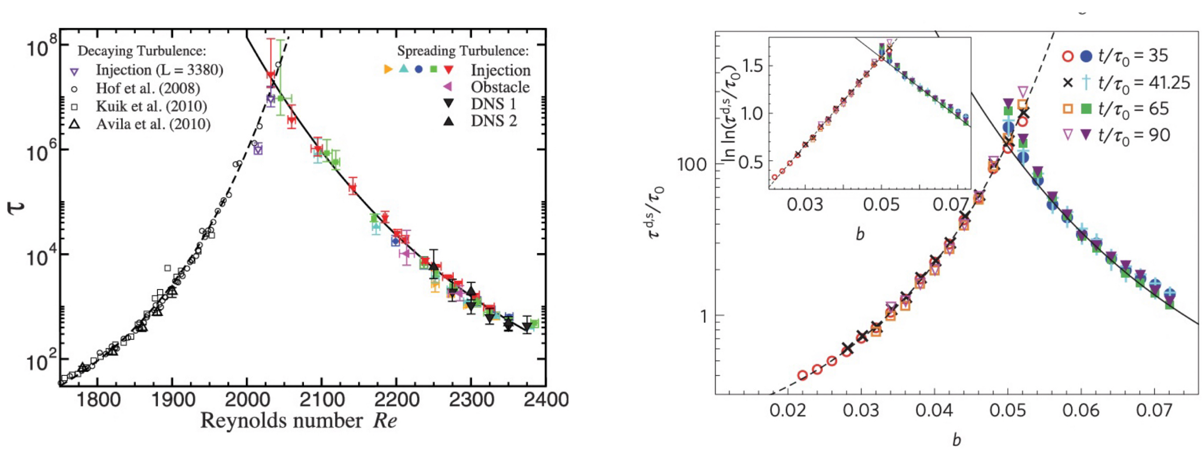

Figure 4: Simulated results for the lifetime of predator-prey populations before ecological collapse and mean time between splitting events of populations (right) compared to the previous data of puff lifetime and puff splitting (left). Both sets of data show exponential scaling in the two effects. (Goldenfeld et al. 2015 [11])

The realization of this interplay is a manifestation of a deeper connection made already in 1989 by Y. Pomeau [12]. Namely, the connection to the out-of-equilibrium universality class of *directed percolation* in statistical physics.
Directed percolation is a continuous phase transition commonly illustrated abstractly on a lattice with occupiable sites (see fig. **??**). An occupied site of this lattice percolates downwards in a directed fashion to another site with a percolation probability p. For a given lattice dimension, the percolation either dies off or spreads to the lower bound of the lattice. A critical probability seperates the two regimes, describing the setup where the site *just* barely percolates to the lower bound. In this sense, directed percolation is a continuous phase transition with an intrinsic notion of temporal evolution (as opposed to static phase transitions, e.g. para-ferro). The evolution of puff behaviour of turbulence or also of population dynamics in predator-prey systems can be represented in a lattice-like way, which illustrates the connection between the three phenomena (see fig. 5).

The scaling behaviour of the occupation density of a lattice with respect to the percolation probability, $\rho \sim (p - p_c)^{\beta}$, can be modeled to arrive at the universal exponent value of $\beta = 0.276$. Looking at the pendant in the transition to turbulence, Hof et al. [13] conducted a measurement of the density of turbulent regions (i.e. the turbulent fraction) for a range of Reynolds numbers in Couette Flow (see fig. 6). They arrived at the best fitted value of $\beta = 0.29 \pm 0.05$ giving quantitative motivation for the placing of the laminar-turbulence transition into the universality class of directed percolation. Further arguments for the connection to directed percolation can be found in papers [13], [14].

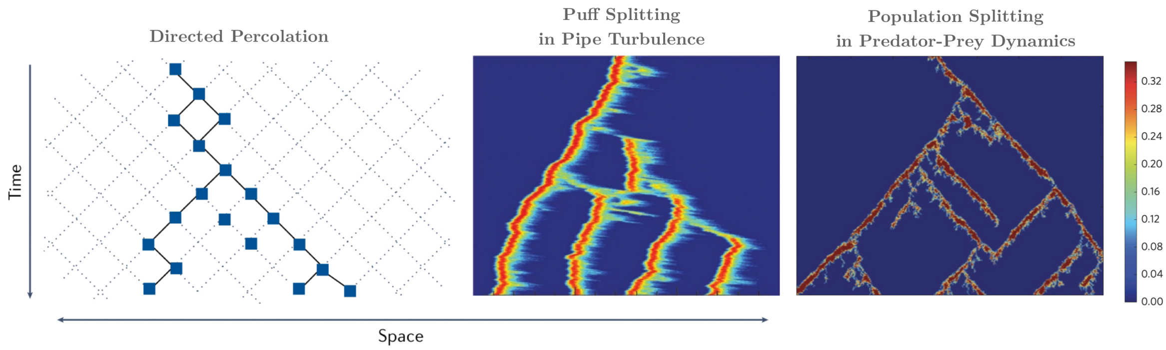

Figure 5: Graphic comparison between the evolution of directed percolation (left), puff splitting (middle) and predator-prey dynamics (right). (Goldenfeld et al. 2015 [11])

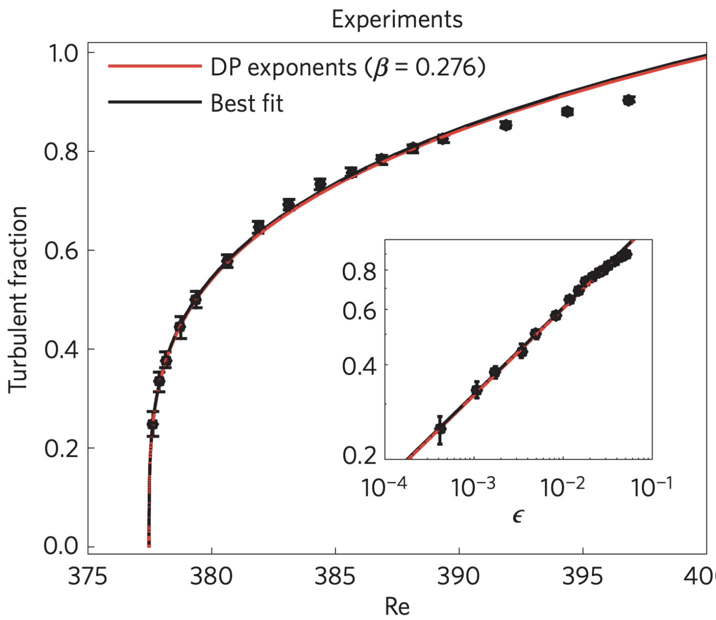

Figure 6: Measurement of the turbulent fraction (i.e. density of turbulent regions among laminar regions) for ca. 2 decades of Re in Couette flow. The resulting scaling exponent was extracted to be $\beta = 0.29 \pm 0.05$, which is close to the exponent $\beta = 0.276$ of directed percolation. (Hof et al. 2015 [13])

4 Conclusion

Turbulence is one of the few phenomena in the mascroscopic classical world observable by eye that is still lacking a conclusive theory ab initio. The strongly interacting and chaotic fluctuating nature of a turbulent flow practically forces us to approach the problem from a statistical standpoint. Such approaches have led to important ideas for fully developed turbulence, like the energy cascade and furthermore the Kolmogorov theory presented in section 2. Kolmogorov theory predicts a scaling law for the energy spectrum of fluctuations, $E(k) \sim \bar{\epsilon}^{2/3} k^{-5/3}$, independent of the large scale geometry of the system at hand hinting at universal structure in turbulence. It turned out, that Kolmogorov's predictions do not agree with experimental results, which is explained by incomplete similarity (scale interference, anomalous scaling) due to intermittency. Hence, the small scale fluctuations do in fact retain memory of the large system scale.
A connection is made to the field of critical phenomena in section 2.4, where the Ginzburg-Landau theory made scaling predictions by coarse-graining over microscopic length scales. Experimental results contradict these predictions, explained by a similar argument of scale interference and a retaining memory of the system's microscopic scale even in large scale fluctuations. The salvation of universality in critical phenomena then came with the introduction of the renormalization group approach by K. Wilson. Hence, there are efforts to find equivalent RG-like approaches to tackle the scale interference in fully developed turbulence.
In section 3, the transition to turbulence was discussed in the context of critical phenomena. Based on experimental work by Hof et al. [9] and theoretical insights of Goldenfeld et al [8], a notion of a critical Reynolds number was introduced. Furthermore, the interplay between puff decay and puff splitting led to a connection to predator-prey dynamics and finally to the universality class of directed percolation. Hence, it could be motivated that even a chaotic, far from equilibrium phenomenon like the transition to turbulence shows universality at onset and that collective dynamics obey simple scaling laws.

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