Abstract

One of the fundamental limits in precision measurements is noise coming from quantum back-action (QBA). This manuscript describes the effects of this noise in greater detail as well as how it can be overcome through quantum non-demolition measurements (QND). Firstly, we describe multiple kinds of possible quantum measurements. Next, we explain how QBA originates and introduce a rigorous model for describing this noise in the case of multiple consecutive measurements. Furthemore, we describe the limiting case of continuous measurements, where the time separation between individual measurements is very small. One feature of this model is a random walk of the continuously measured observables due to the effects of QBA. Lastly, we present one example of a recent QND measurement to see how the described techniques can be implemented in practice.

Contents

1 Introduction

Precise measurement of observables has been the backbone of many contemporary experiments such as gravitational wave detectors. At a sufficiently small scale, quantum effects start to play a prominent role and influence the achievable measurement precision. The exact mechanism of those quantum effects function is nevertheless often not adequately explained. For this reason, this manuscript reviews the precise workings of said quantum effects and ways of overcoming them.

The term *quantum back-action* (QBA) has been coined to describe the quantum effects that disturb our measurement accuracy. More accurately, it describes the way each measurement influences the system. Should we conduct repeated measurements, this introduces a noise that fundamentally limits the achievable precision. For this reason, we want to do quantum non-demolition (QND) measurements, which do not disturb the system and allow us to measure repeatedly. We will examine QBA from different aspects to be able to outline possible QND methods.

As an example of a very common system where QBA plays a role let us consider position measuring interferometers. Such devices have been used in the aforementioned gravitational wave detectors. They contain a mirror, which reflects photons. If laser strength is increased, more photons are emitted and signal-to-noise ratio improves. On the other hand, light pressure on the mirror increases, which constitutes QBA and worsens the accuracy. Thus we can achieve ideal accuracy by configuring the laser strength to a certain intensity – the sensitivity depending on input power is displayed in Fig. 1.

This is a principal example of QBA, which should serve as intuition. Nevertheless, we can already make an important observation. Namely that in this configuration, one cannot achieve a greater precision than the so called standard quantum limit (SQL). Luckily, SQL can be overcome if we somehow reduce QBA in a so-called *back-action evading measurement* (BAE).

In the next section 2, we will detail which sorts of quantum measurements we may even hope to carry out. Then, we will focus on the process of quantum measurement in an even greater detail: in section 3 we will present an intuitive, but not very rigorous picture, which will be followed by a more rigorous description of quantum measurements in section 4. Section 3 will serve to explain mechanisms behind the SQL and section 4

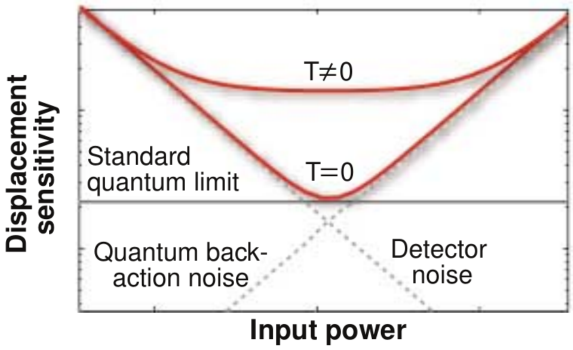

Figure 1: The relationship between displacement sensitivity and laser power in an interferometer; we observe two competing effects: increasing laser strength improves signal-to-noise ratio, but on the other hand introduces quantum back action. Taken from [1].

will also describe a process of many successive measurements and the limit of continous measurements. Lastly, section 5 will introduce an example of a QND measurement in practice.

2 Quantum measurement typology

Let us imagine a common situation, in which quantum effects play a role: we measure a classical force $F(t)$ weakly coupled to a quantum linear harmonic oscillator. The quantum oscillator which we want to measure has canonical variables $\hat{x}$ and $\hat{p}$. They are in some way interlinked: measuring one has an effect on the other. We will outline the possible ways of measuring both variables. A more detailed overview of this problematic is presented in [2].

The Hamiltonian of our system is

$$\hat{H}_0 = \hat{p}^2/2m + m\omega^2\hat{x}^2/2\,. \tag{1}$$

with ω being the frequency, m mass and $\hat{x}$ with $\hat{p}$ canonical position and momentum operator. As we know, $\hat{x}$ and $\hat{p}$ do not commute, which is the source of QBA.

For convenience, we define the following new variables:

$$\hat{X}_1(\hat{x},\hat{p}) = \hat{x}\cos\omega t - (\hat{p}/m\omega)\sin\omega t\,, \qquad \hat{X}_2(\hat{x},\hat{p}) = \hat{x}\sin\omega t + (\hat{p}/m\omega)\cos\omega t\,. \tag{2}$$

The new variables additionally fullfill the following equation:

$$\hat{x} + i\hat{p}/m\omega = (\hat{X}_1 + i\hat{X}_2)e^{-i\omega t}\,. \tag{3}$$

It is best to visualise x and p as 2D coordinates. Then, eq. 3 tells us that the new coordinates are rotating clockwise relative to coordinates $(x, ip/m\omega)$. Furthemore, if one writes out the Heisenberg equation of motion for both of the new coordinates, the result is that both X_1 and X_2 stay constant in time:

$$\frac{\mathrm{d}\hat{X}_{1,2}}{\mathrm{d}t} = \frac{\partial\hat{X}_{1,2}}{\partial t} - \frac{i}{\hbar}\left[\hat{X}_{1,2}, \hat{H}_0\right] = 0\,. \tag{4}$$

Let us also remind the reader that the partial derivative of $\hat{X}_{1,2}$ is in fact non-zero. Only the full derivative operator of $\hat{X}_{1,2}$ is zero and this is the reason for choosing such a reference frame. Such a model then easily generalises to different contexts.

Next, we can in principle do three types of measurements, which are depicted in Fig. 2. Firstly, we can do a so-called *amplitude and phase* measurement. This corresponds to measuring both X_1 and X_2. In the case of the smallest possible errors, the uncertainties still do not disappear, so the measurement result is a circle. This is the standard and least sophisticated way to measure observables. The minimum uncertainty circle is caused by back-action and the lowest achievable uncertainty is the SQL.

Secondly, we can do *quantum counting*, where we measure the quantity $(\hat{X}_1^2 + \hat{X}_2^2)$. This corresponds to measuring the number of quanta[1] N in the oscillator while ignoring the phase. Thus the outcome of the measurement is a circle in the plot of X_1 and X_2. One problem with this type of measurement is that it cannot register forces corresponding to less than one quantum − this is another fundamental limit, that cannot be

[1] In fact the occupation number operator can be expressed using our rotating coordinates as $\hat{N} = m\omega/(\hbar 2)(\hat{X}_1^2 + \hat{X}_2^2) - 1/2$.

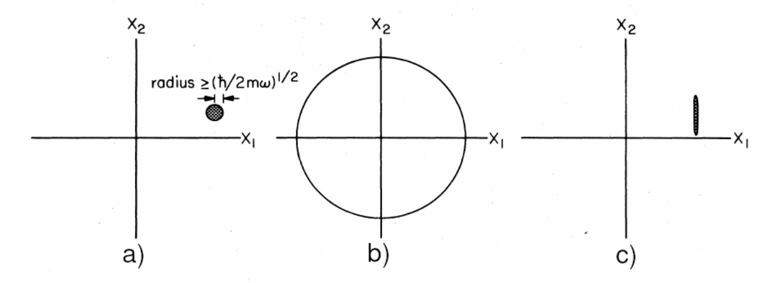

Figure 2: Various modes of measurement for two conjugate variables. a) Amplitude and phase measurement, where we measure both coordinates X_1 and X_2. In the case of minimum errors, we obtain an error circle. b) Photon counting, where we measure N, but ignore phase. The result is a circle. c) BAE measurement, where we minimise the uncertainty of X_1 at the expense of X_2. Taken from [1].

overcome. Also, the occupation number is usually measured in a destructive way. A photodiode, for example, destroys photons upon measuring them. This does not allow us to make multiple measurements to refine the accuracy.

Lastly, we can do *BAE measurements*. In this case we measure X_1 but through a clever mechanism direct the back-action to X_2. Thus the uncertainty of X_1 is minimised, while the uncertainty of X_2 becomes arbitrarily high. This is not a problem, as we only care[2] for X_1. In principle, arbitrarily high precision can be achieved with this method. In practice, for high precisions the assumption that $F(t)$ is classical stops being valid. Thus we arrive at another limit. This limit is luckily negligible in practice.

Let us remind the reader that Fig. 2 only outlines what is theoretically possible to measure and it does not tell us in any way how to realize such a measurement. For example we wrote that usually, $\hat{N}$ is measured in a destructive way by photodetectors, but there might be a non-destructive way. Also, we mentioned that in the case of BAE, we somehow redirect the back-action from X_1 to X_2, but we did not propose mechanisms how to do that. Some of such mechanisms can be found in [2], but for each concrete case, there might be some other practical limit that limits the accuracy of X_1. Nevertheless, BAE measurements offer the highest theoretically achievable accuracy. This is desired for certain applications, where we already approach SQL.

Another thing to keep in mind is that Fig. 2 only displays one measurement. In some applications, we want to measure the system repeatedly: in such cases even a small error can become amplified. This is what we will focus on in the upcoming sections.

3 Conventional quantum-mechanical measurements

In this section, we remind the reader of the effects of conventional quantum mechanical measurements on states. To this end, let us consider a well-known quantum system: a free particle in a state $|\psi\rangle$ with a wave function of a Gaussian wave packet:

$$\langle x|\psi\rangle = \frac{1}{(\pi\sigma)^{1/4}} \exp\left[\frac{(x-\mu)^2}{2\sigma}\right] . \tag{5}$$

Here, μ is the *mean position* and σ is the *half-width* of the wave packet. The half-width σ represents our uncertainty: the wider the wave packet, the bigger our uncertainty and vice versa. If we do not disturb the particle, it will evolve in time. If it had one concrete momentum value, it would just move in one direction. In quantum mechanics though, its momentum is uncertain.[3] Thus we can imagine the particle has "multiple momenta at once" and each momentum value causes the particle to move with a different velocity. As an effect of this, the wave packet gets broader and broader with time, the half-width σ increases as can be seen in Fig. 3.

As it happens, the particle does not get to evolve freely forever, it will eventually interact with its environment. One form of this interaction is our *measurement*. According to the Copenhagen interpretation, when we conduct a measurement of an an *observable* $\hat{O}$, the state collapses into an eigenstate of the observable. It would be typical to measure the particle position x, so let us consider the observable $\hat{x}$. Ideally, after measuring $\hat{x}$, we would know the particle position x. We would furthermore get x randomly according to the (Gaussian) distribution $P_{\hat{x}}(x) = |\langle x|\psi\rangle|^2$.

In many cases, it is reasonable to assume that we can obtain one precise value of x. This however is not experimentally achievable, as we always measure the value with some error bars. Or more precisely, we obtain

[2]For example, in the case of an interferometer we only care about the position, but not about the momentum: we can afford to ignore one canonical variable.

[3]In fact, the so called momentum representation $\langle p|\psi\rangle$ is also a Gaussian.

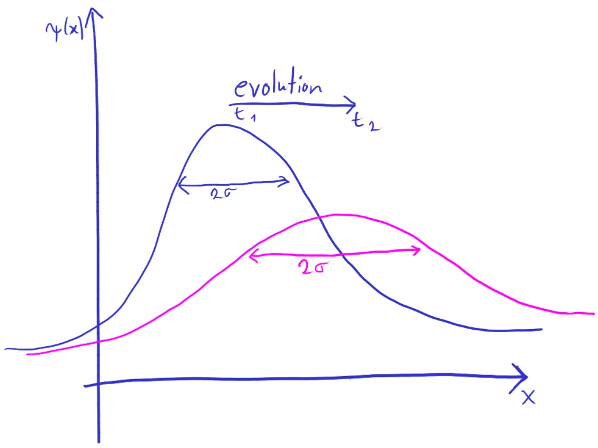

Figure 3: Free evolution of a Gaussian wavepacket. Its mean value moves uniformly in one direction (according to its group velocity) while its half-length increases.

a probability distribution for x, e.g. $P_{\hat{x}'}(x)$. As this probability distribution is in most cases Gaussian, we again get

$$P_{\hat{x}'}(x) = |\langle x|\psi'\rangle|^2 = \text{Gaussian}\,. \tag{6}$$

Here, $|\psi'\rangle$ denotes the new, post-measurement state. Because the result of our measurement is not one value x, but a probability distribution $P_{\hat{x}'}(x)$, we can reconstruct $\langle x|\psi'\rangle$ with some new mean μ' and most notably width σ'. It will still be Gaussian, as the square root of a Gaussian is still a Gaussian.

$$\langle x|\psi'\rangle = \frac{1}{(\pi\sigma')^{1/4}} \exp\left[\frac{(x-\mu')^2}{2\sigma'}\right]. \tag{7}$$

We could say we have not conducted an *ideal measurement*, but an *imperfect measurement*. In this formalism, the ideal measurement would be the limit $\sigma' \to 0$. The whole imperfect measurement process can be seen in Fig. 4.

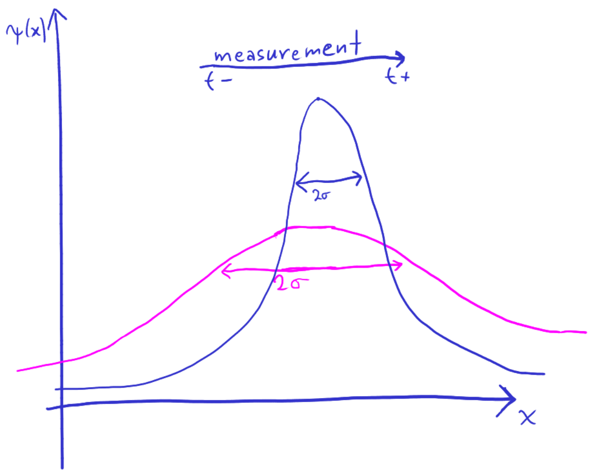

Figure 4: Imperfect measurement of a Gaussian wavepacket in time t. Time $t-$ is the state just before measurement and $t+$ just after.

To summarize, in our toy model[4] we observe a particle in a state of a Gaussian wave packet with half-width σ. This width is indicative of our uncertainty and when we let the particle evolve freely, it increases. On the other hande, measuring the particle's position, we "reset" the width back to a certain σ_R as depected in Fig. 5.

3.1 Effect on conjugate observables

So far we have looked at how the measurement of position influences position. As we know, there is also a canonical conjugate observable $\hat{p}$, the momentum. Due to quantum-mechanical effects, the measurement of

[4]The model is nevertheless indicative of a broader scope, as many observables tend to be Gaussian.

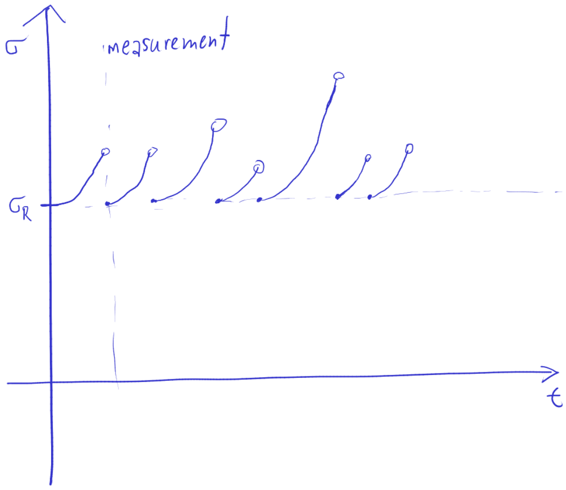

Figure 5: The width of a Gaussian wave packet in time as we make measurements and let the state evolve. During free evolution the half-width gradually inreases, the measurement process then resets the width back to a certain value σ_R.

$\hat{x}$ will disturb $\hat{p}$ and vice versa. This disturbance is quantified by the Heisenberg uncertainty relation, which relates the minimum uncertainty of x and p denoted as Δx and Δp:

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4} \Leftrightarrow (\Delta x)(\Delta p) \geq \frac{\hbar}{2}. \tag{8}$$

Now let us consider the case where we measure $\hat{x}$ with uncertainty Δx: this uncertainty can be imagined as the half-width of our wave packet. Then, the uncertainty of $\hat{p}$ will be $p_u = \hbar/(2\Delta x)$. This means that the wave packet has some unknown momentum p', which we can characterise[5] with the uncertainty p_u. If we conduct another measurement after time T, the whole wave packet will move by Tp/m. As we do not know the value of p, an additional uncertainty Tp_u/m is created. If we conduct the measurements repeatedly, we might expect the particle position to undergo a random walk: this will be shown more rigorously in further sections. It is precisely this effect that we call QBA and that we mentioned in section 1.

On the other hand, measuring $\hat{p}$ shifts the position, which does not affect the momentum. We can measure $\hat{p}$ repeatedly without any back-action. Thus we call $\hat{p}$ a *QND observable*.[6] In this nature, we can define a *QND measurement* of an observable $\hat{A}$ a sequence of measurements of $\hat{A}$ such that the result of each measurement is completely predictable from the result of the first measurement and other information about the initial state of the system [3].

Why do we care about QND measurements? If the result of our measurement is random, as is the case in a non-QND measurement, our accuracy is diminished. The randomness effectively becomes noise. To quantify the noise, we turn to Heisenberg uncertainty relation and consider the case of equal and minimal uncertainties.[7] This is the standard quantum limit, which we mentioned in setion 1: a naturally occuring "barrier" of minimal uncertainty.

4 Continuous measurements

Let us now formalise the results of the previous section as is outlined in the paper [4]. There, a mathematical formalism is developed for describing imperfect measurements in terms of so called *operations* and *effects*. Then, "a sequence of n [...] instantenous measurements of position separated by time τ" is conducted with the intention of "determining the system evolution for the limit $n \to \infty$ and $\tau \to 0$." Since we consider the limiting case, we call the resulting evolution an evolution with *continuous measurements*.[8] This can be viewed as a model of the case where we naively measure $\hat{x}$ periodically without managing the back-action. We can then quantify the back-action effects and understand them more.

[5]This characterisation is only partial: for if we knew the exact momentum, there would be no uncertainty.

[6]Momentum is not a QND observable in all systems though. For instance, in the case of a linear harmonic oscillator, measuring $\hat{x}$ disturbs momentum. If we imagine a spring, the position influences the tautness of the spring, which changes the velocity.

[7]It is possible to intentionally create states with one uncertainty being small and the other large for the purpose to only measure one observable. Such states are called *squeezed states* and will not be discussed here.

[8]The measurements are not continuous in a true sense though, we merely consider our characteristic time of interest to be larger than τ.

What are the potential benefits of continuous measurement? Firstly, we might want to extract information continually if we want to have a good time resolution. For a second argument, consider again a free particle. In the continuous measurement process, the evolution of its half-width σ in time is of interest. If we look at the graph in Fig. 5 and conduct the limit of $\tau \to 0$, we see that $\sigma(t) \approx \sigma_R = \text{const.}$: the half-width does not increase with time, which is a good thing.

Let us lastly mention one caveat: we cannot simply take the limit $\tau \to 0$, we have to compensate for it somehow. We will explain this compensation in detail in section 4.2, for the moment we just wanted to emphasize that the process of continuous measurement is in some sense non-trival. In fact, it combines free evolution and measurement into one process.

4.1 Repeated imperfect measurements

We will now take a look at a concrete model from [4], where repeated quantum measurement will be implemented. The model consists of a physical subsystem and a system of *meters*, i. e. measuring apparatuses. The meter has canonical variables $\{\hat{X}_r, \hat{P}_r\}_{r=1}^n$. Those coordinates may be regarded as the outcomes of n repeated position measurements on the subsystem, which has its own canonical variables $\hat{x}$, $\hat{p}$.

The coupling between the meters and the subsystem is given by the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \sum_{r=1}^{n} \delta(t - r\tau)\hat{x}\hat{P}_r \,. \tag{9}$$

Here, $\hat{H}_0$ can be an arbitrary Hamiltonian, which describes the studied subsystem. The second term describes the interaction that is caused by a measurement: the Dirac delta suggests an instantaneous interaction and $\hat{x}\hat{P}_r$ describes how measuring the position gives a momentum kick to the meter.

We work in a joint Hilbert space $\mathcal{H}_m \otimes \mathcal{H}_0$, where $\mathcal{H}_m$ is the meter space and $\mathcal{H}_0$ is the subsystem space. We prepare the r-th meter in a pure Gaussian state $|M_r\rangle$ such that it has the following wave function:

$$\langle X_r|M_r\rangle \equiv M_r(X_r) = (\pi\sigma)^{-1/4} \exp\left(-\frac{X_r^2}{2\sigma_R}\right) \tag{10}$$

for some σ_R (we may suppress the argument X_r when the context is obvious). In a similar spirit, we define the *meter operator*[9]

$$\hat{M}_r(X_r) \equiv \langle X_r|e^{-i\hat{x}\hat{P}_r/\hbar}|M_r\rangle = (\pi\sigma_R)^{1/4} \exp\left(\frac{-(\hat{x} - X_r)^2}{2\sigma_R}\right) \tag{11}$$

$$= \int \mathrm{d}x\, (\pi\sigma_R)^{1/4} \exp\left(\frac{-(\hat{x} - X_r)^2}{2\sigma_R}\right) |x\rangle\langle x| = \int \mathrm{d}x\, (\pi\sigma_R)^{1/4} \exp\left(\frac{-(x - X_r)^2}{2\sigma_R}\right) |x\rangle\langle x| \,. \tag{12}$$

We obtain the third expression in eq. 11 by applying an $\hat{x}$-translation operator $e^{-i\hat{x}\hat{P}_r}$ to eq. 10. This is an intuitive way to understand the meter operator, we include the other expressions in eq. 12 just for further context. We might even write, abusing our notation somewhat, the following:

$$\hat{M}(X_r) = M(X_r - \hat{x}) \,. \tag{13}$$

Having described the meter, let us denote $\hat{\rho}(t) \in \mathcal{H}_0$ the state of the measured subsystem with $t = t_r-$ being the time just before the r-th measurement and t_r+ just after. The state of the whole system just before the r-th measurement will be:

$$\rho_{0m}(t_r-) = |M_r\rangle\langle M_r| \otimes \rho(t_r-) \,. \tag{14}$$

Now let us look how the state will look like after a measurement of $\hat{X}_r$, which is described by the interaction term in the Hamiltonian from eq. 9:

$$\rho_{0m}(t_r+) = e^{-i\hat{x}\hat{P}_r/\hbar}|M_r\rangle\langle M_r| \otimes \rho(t_r-) e^{i\hat{x}\hat{P}_r/\hbar} \,. \tag{15}$$

The only change is the application of the shift operators $e^{i\hat{x}\hat{P}_r/\hbar}$. Let us take a look at the probability distribution of X_r:

$$P_{\hat{X}_r}(X_r) = \mathrm{tr}\left(|X_r\rangle\langle X_r|\rho(t_r+)\right) = \mathrm{tr}\left(|X_r\rangle\langle X_r|(e^{-i\hat{x}\hat{P}_r/\hbar}|M_r\rangle\langle M_r| \otimes \rho(t_r-) e^{i\hat{x}\hat{P}_r/\hbar})\right) \tag{16}$$

$$= \mathrm{tr}\left(\langle X_r|e^{-i\hat{x}\hat{P}_r/\hbar}|M_r\rangle\rho(t_r-)\langle M_r|e^{i\hat{x}\hat{P}_r/\hbar}|X_r\rangle\right) = \mathrm{tr}\left(\hat{M}(X_r)\rho(t_r-)\hat{M}^+(X_r)\right) \,. \tag{17}$$

[9] In [4] it is referred to as the system operator. We found this name to be more illustrative.

The after-measurement state of the studied subsystem will be (given outcome X_r):

$$\hat{\rho}(t_r+) = \frac{\langle X_r|\rho_{0m}(t_r+)|X_r\rangle}{P_{\hat{X}_r}(X_r)} = \frac{\hat{M}_r(X_r)\hat{\rho}(t_r-)\hat{M}_r^+(X_r)}{P_{\hat{X}_r}(X_r)}\,. \tag{18}$$

Thus, the subsystem after measurement is determined by the meter operator $\hat{M}_r$. An important part of the meter operator is the parameter σ_R. Now it represents a different quantity from when it appeared section 3. Namely, it represents the measurement precision: if $\sigma_R \to 0$, we have an ideal measurement, otherwise we are dealing with an imperfect measurement. We can influence this parameter by our choice of a measuring apparatus.

Having described the measurement process, free evolution is managed simply by the evolution operator

$$\hat{U}(\tau) = \mathrm{id} \otimes e^{-i\hat{H}_0\tau/\hbar}\,. \tag{19}$$

Thus we can express the state of our subsystem in a time just after a general n-th measurement given previous outcomes $\{X_r\}$:

$$\hat{\rho}(\{X_r\}, t_n+) = \frac{\left[\prod_{i=1}^n \hat{M}_r(X_r)\hat{U}(\tau)\right]\hat{\rho}(0)\left[\prod_{i=1}^n \hat{M}_r(X_r)\hat{U}(\tau)\right]^+}{P_{\{\hat{X}_r\}}(\{X_r\})}\,. \tag{20}$$

The probability of measuring the specific $\{X_r\}$ is given by

$$P_{\{\hat{X}_r\}}(\{X_r\}) = \mathrm{tr}\left[\left[\prod_{i=1}^n \hat{M}_r(X_r)\hat{U}(\tau)\right]^+\left[\prod_{i=1}^n \hat{M}_r(X_r)\hat{U}(\tau)\right]\hat{\rho}(0)\right]\,. \tag{21}$$

This formula might seem intimidating, but it in fact just describe the occuring process: first we let the state evolve, then measure it, then let evolve, then measure etc. The meaning of eq. 20 will become more clear for the case of $n = 1$:

$$\hat{\rho}(X_1, t_1+) = \frac{\hat{M}_1(X_1)\hat{U}(\tau)\hat{\rho}(0)\hat{U}^+(\tau)\hat{M}_1^+(X_r)}{P_{\hat{X}_1}(X_1)}\,. \tag{22}$$

We first start with initial $\hat{\rho}(0)$. Then the free evolution is described by the pair $\hat{U}(\tau)$, $\hat{U}^+(\tau)$ per eq. 19. After that we measure, which modifies the state as in eq. 18. For the general case of n measurements, we just repeat this.

Thus we have obtained a powerful formalism for repeated measurements, which can be applied to many different kinds of subsystems.

4.2 Thought experiment

In the next subsection, we will apply our formalism on a free particle. Before that let us conduct a thought experiment regarding the limit of continuous measurement. Our main concern is the following: quantum measurement usually disturbs the quantum state. If we measure more and more frequently, the disturbance will increase. So how do we measure in the continous limit, but keep the disturbance constant?

We will approach this from an information-theoretical standpoint. We make the claim that it is *information retrieval* what disturbs the quantum state. As is written in [3]: “It matters not how the information is extracted, nor where it is stored in: a person’s brain, on magnetic tape, or in some minute change of the state of some other particle. So long as the information exists somewhere in the universe outside the original particle (more precisely, ‘outside the particle’s wave function’), future measurements of the particle will reveal that the disturbance has occurred. The only way to undo the disturbance is to ‘run the measuring apparatus perfectly backward’ and thereby reinsert all the information back into the particle. Only if no trace of the information remains anywhere, not even in the experimenter’s brain, can the particle return to its original undisturbed state.”

Thus, let us try to quantify the information retrival. We imagine again the case of a Gaussian free particle. Imagine that each position measurement gives us m bits of information about the system as a “tradeoff” for certain disturbance. If we want to maintain the disturbance while increasing measurement frequency $1/\tau$, we must sacrifice the amount of bits per measurement.

It is reasonable to assume that $m \propto \sigma_R^{-1}$, where σ_R now is not the half-width, but the meter precision, as introduced in eq. 10. Now, if we want to obtain M bits of information over some time period T, it must hold that

$$M = mn = m\frac{T}{\tau} \Rightarrow \frac{M}{T} \propto \frac{1}{\sigma_R\tau}\,. \tag{23}$$

From this heuristic argument it follows the information retrieval rate M/T is proportional to $\sigma_R\tau$. We have argued that this information retrieval rate should also be tied to the disturbance rate. This is why we define the parameter $D \equiv \sigma_R\tau$ and in the limit $\tau \to 0$ demand that it stays constant: If we did not do it, then the information retrieval rate would go to infinity and this is unphysical. Moreover, the effects of the measurement on the system would also spiral oout of control, which we do not want.

The reasoning presented in this subsection is perhaps very heuristic. It nevertheless provides an intuition for the phenomenon of continuous measurements. Moreover, the result (that D must stay constant) will be again proven rigorously in the following section through independent means.

4.3 Free particle

In this subsection we pick up the results obtained in section 4.1 and apply them on the case of a Gaussian free particle as was done in [4]. The free particle has the Hamiltonian $H_0 = p^2/(2m)$. It can be shown, that its most general pure state has the wave function [5, p. 339]:

$$\langle x|\psi(t)\rangle = e^{i\varphi(t)}\left(\pi\Delta(t)\right)^{-1/4}\exp\left(-\frac{1-i\epsilon(t)}{2\Delta(t)}(x-a(t))^2+\frac{i}{\hbar}b(t)x\right). \tag{24}$$

This is a generalisation of eq. 5. Here, $\varphi(t)$ is an unimportant phase, latin letters denote the *expectation values* of observables and greek letters denote their *uncertainties.* Most importantly, $\Delta(t)$ is the width of the wave packet. More precisely:

$$\langle\hat{x}\rangle = a(t)\,, \qquad \langle\hat{p}\rangle = b(t)\,, \tag{25}$$

$$\langle(\Delta\hat{x})^2\rangle = \Delta(t)/2\,, \qquad \langle(\Delta\hat{p})^2\rangle = \frac{1+\epsilon^2(t)}{\Delta}\frac{\hbar^2}{2}\,. \tag{26}$$

We call the latin parameters *first order* parameters and the greek parameters *second order.* The evolution of our state is completely determined by the evolution of those parameters.

First, let us look at unitary (free) evolution. Average momentum does not undergo a change, average position changes linearly as the wave packet travels with a velocity b/m. Let us again consider the situation, where we measured the system in time t_{r-1}, and are interested in the value at time $t_r = t_{r-1}+\tau$:

$$a(t_r-) = a(t_{r-1}+) + b\tau/m\,, \qquad b(t_r-) = b(t_{r-1}+)\,. \tag{27}$$

The second order parameters have the following unitary evolution:

$$\Delta(t_r-) = \Delta(t_{r-1}+) + \frac{1+\epsilon^2(t_{r-1}+)^2}{\Delta(t_{r-1}+)}\left(\frac{\hbar\tau}{m}\right)^2 + 2\epsilon(t_{r-1})\frac{\hbar\tau}{m}\,, \tag{28}$$

$$\epsilon(t_r-) = \epsilon(t_{r-1}+) + \frac{1+\epsilon^2(t_{r-1}+)}{\Delta(t_{r-1}+)}\frac{\hbar\tau}{m}\,. \tag{29}$$

We can say that they quadratically increase with time.

The density operator of our pure state is of course $|\psi(t)\rangle\langle\psi(t)|$, so we can apply eq. 18 to get the evolution after measurement:

$$|\psi(t_r+)\rangle = \frac{\hat{M}(X_r)|\psi(t_r-)\rangle}{P^{1/2}_{\hat{X}_r}(X_r)}\,. \tag{30}$$

The meter operator is a Gaussian and the product of two Gaussians is still a Gaussian. Thus the after-measurement function will also be a Gaussian, only with the different parameters. The parameters can be expressed as:

$$a(t_r+) = a(t_r-) + \frac{C_r-1}{C_r}(X_r - a(t_r-))\,, \tag{31}$$

$$b(t_r+) = b(t_r-) + \hbar\frac{\epsilon(t_r-)}{\Delta(t_r-)}\frac{C_r-1}{C_r}(X_r - a(t_r-))\,, \tag{32}$$

$$\Delta(t_r+) = \Delta(t_r-)/C_r\,, \qquad \epsilon(t_r+) = \epsilon(t_r-)/C_r\,. \tag{33}$$

with $C_r = 1 + \Delta(t_r-)/\sigma \geq 1$ being the contraction factor. Notice that the observed value X_r is random, thus the evolution of a and b will also proceed in random "jumps". Also, the measured X_r depends on the mean position of the particle, so the expression $(X_r - a(t_r-))$ is translation-invariant: it represents the deviation of the measured X_r from the mean position.

On the other hand, the second order parameters Δ and ϵ evolve predictably, they strictly decrease as $C_r \geq 1$.

One can then tune the measurement interval τ and precision σ_r such that the second order parameters do not change after each measurement. It is indeed possible to arrive at such a stationary configuration and it can even be proven that this configuration is even stable under perturbations. It is practical to choose this stationary configuration if we want to do multiple measurements, so that the second order parameters do not go to zero nor infinity.

Furthemore, if we focus on the evolution of mean momentum from $t_{r-1}+$ to t_r+ (eq. 27 and 33), we observe that it undergoes a brownian random walk. This will lead to the momentum value increasing without bound. This is purely the effect of QBA, we see that in this model it is truly severe. We might include some dissipation into the system to make it more realistic, though still the momentum will undergo a random walk, which creates noise. We observe similar results for a general quantum system: it can be shown that momentum diffusion will be superposed on the intrinsic dynamics of the system. A similar statement can be made about the position, only its random walk will not be brownian. Rather, it will be the integral of brownian motion.

Lastly let us talk about the limit of continuous measurements. In this limit, we send τ to zero and (after some calculations) get

$$\Delta \to \Delta_c = \left(\frac{2\hbar}{m}\right)^{1/2} D^{1/2}\,, \tag{34}$$

$$\epsilon \to \epsilon_c = 1\,. \tag{35}$$

We see that Δ depends on the parameter $D = \sigma_R \tau$. Thus, if we want it to stay finite, we need to keep D constant, or in other words send $\sigma_R \to \infty$. We have thus confirmed the heuristic result from section 4.2.

5 QND measurements in practice

In this section we summarise the paper [6] as a concrete example of utilising QND to yield better measurement accuracy. In this experiment, the aim is to measure the position $\hat{x}$ of a harmonic oscillator. Usually, position measurement is done via an interferometric setup. However, "The ultimate sensitivity of an interferometer depends on the back-action that photons exert onto the mechanically compliant mirror, caused by radiation pressure," as is described in [1]. The noise from QBA thus constitutes a certain "minimum" measurement imprecision, the SQL. With QND we evade QBA and thus can surpass the SQL.

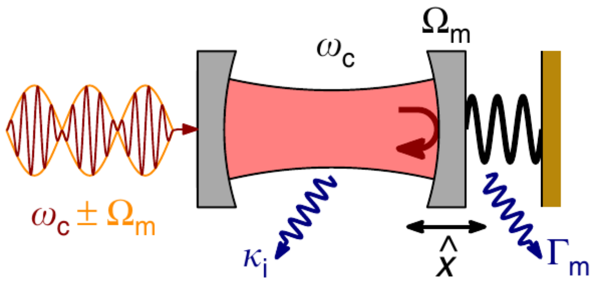

Figure 6: A diagram of the optomechanical experimental setup. Taken from [6].

The experimental setup is depicted in Fig. 6. It consists of a harmonic oscillator with frequency Ω_m coupled to an optical cavity with frequency ω_c. The usefulness of this fact lies in the fact that the quantum harmonic oscillator can be coupled to a classical force $F(t)$. Measuring the parameters of the oscillator can then tell us about $F(t)$. In this though, there is no force.

The mechanical oscillator is coupled to the cavity in a simple manner: it influences the length of the cavity. Thus it influences the admissible wavelengths of photons in the cavity, or in other words their frequencies. To gather information about the system we probe the cavity with two lasers at the frequency $\omega_c \pm (\Omega_m + \delta)$, where δ is a parameter called the *detuning*.

What we measure is the photocurrent *power spectral density*, for which there is the following theoretical relationship:

$$\overline{S}(\omega) = 1 + \eta\Gamma_{\text{eff}}^2 C\left[\overline{n}|\chi_m(\omega-\delta)|^2 + (\overline{n}+1)|\chi_m(\omega+\delta)|^2 + C|\chi_m(\omega-\delta) - \chi_m(\omega+\delta)|^2\right]\,. \tag{36}$$

Here, η is the overall detection efficiency, χ_m is the mechanical susceptibility of the oscillator with total mechanical linewidth Γ_{eff}. More importantly, $\overline{n}$ is the mean occupation number of the mechanical oscillator (e. g. the signal we want to measure) and C is a constant called optomechanical cooperativity, which is proportional to the input power. The power spectral density corresponds to the occupation number operator $\hat{N}$ of the system and by only measuring it, we relinquish the information about the phase. Thus, BAE measurements are in principle possible per the analysis in sec. 2, Fig. 2.

Returning to eq. 36, the terms $\overline{n}|\chi_m(\omega-\delta)|^2$ and $(\overline{n}+1)|\chi_m(m+\delta)|^2$ are the *Stokes* and *anti-Stokes* motional sidebands. The term $C|\chi_m(\omega-\delta)-\chi_m(\omega+\delta)|^2$ describes the effects of QBA. We see that if we choose $\delta = 0$, the QBA noise becomes zero. The power spectral density is depicted in Fig. 7, where the back-action noise can be seen in orange. As we mentioned, for $\delta = 0$ the noise disappears and also the Stokes and anti-Stokes sidebands merge into one.

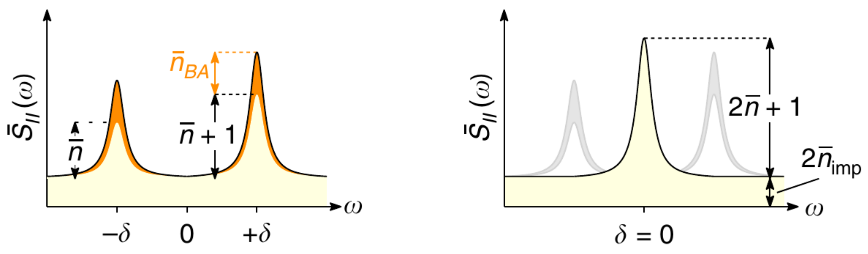

Figure 7: Observed power spectral density from eq. 36. In the left picture we observe two sidebands and a QBA noise contribution in orange. In the right picture we have set $\delta = 0$, thus eliminating QBA and uniting the two sidebands. Taken from [6].

One might think that setting $\delta = 0$ will simply remove QBA allowing us to have unlimited precision. The measurement however still has some practical limitations, which we will now outline. The power spectral density can be shown to consist of the following components:

$$S(\omega) \propto \overline{n}_{\text{imp}} + \overline{n}_{\text{BA}} + \left(\overline{n} + \frac{1}{2}\right) . \tag{37}$$

Firstly, $\overline{n}$ is the signal we want to measure and the other two sources are noise. We observe shot noise[10] $\overline{n}_{\text{imp}} = (8\eta C)^{-1}$. It is inversely proportional to the signal intensity C, which is to be expected because higher C means better signal-to-noise ratio. Increasing C however adds to the heat generated by back-action, which is the second term. Heat back-action is not QBA, but rather heating from optical absorption; it has the following form $\overline{n}^{\text{h}}_{\text{BA}} = \beta C$, where β is a coefficient dependent on the efficiency of the setup. The minimum total added noise (together with shot noise) is $\overline{n}^{\text{BAE}}_{\text{tot}} = (2\eta/\beta)^{-1/2}$.

A diagram of the observed components of $S(\omega)$ is presented in Fig. 8. It bears resemblance to Fig. 1 with the difference that Fig. 1 presents the non-BAE case.

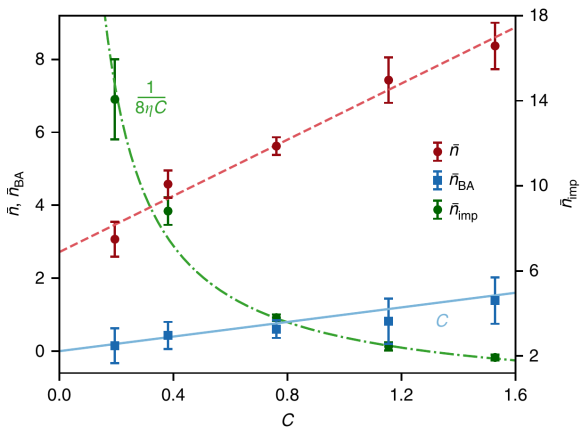

Figure 8: The observed power spectral density is proportional to a sum of various occupation numbers n. This graph presents the individual components: $\overline{n}$ is the occupation number of the observed oscillator, the signal. $\overline{n}_{\text{imp}}$ is the shot noise and $\overline{n}_{\text{BA}}$ is the noise due to heating back-action. Taken from [6].

While the experiment only consisted of the BAE measurement, we could also do a traditional, amplitude-and-phase measurement. It would still have have shot noise $\overline{n}^{\text{hom}}_{\text{imp}}$, but additionally a QBA noise $\overline{n}_{\text{BA}}$. The

[10] We have already seen shot noise in Fig. 1 as "Detector noise".

Heisenberg uncertainty relation gives us a relation between those two noises in the *optimal* case: $4\sqrt{\overline{n}_{\mathrm{imp}}\overline{n}_{\mathrm{BA}}} \geq 1$ We can minimise this noise to get to the SQL level, which yields the total noise $\overline{n}_{\mathrm{tot}}^{\mathrm{SQL}} = (4\eta)^{-1/2}$. As we can see, BAE measurements yield better accuracy than this for $\beta < 1/2$.

In the experiment, back-action evading was observed; the back-action noise was in some cases as high as 14 %. Nevertheless, the experimenters were not able to beat SQL, the BAE noise was 2.78 times the optimal SQL noise. This means that although BAE is possible, there is still work to be done if we want to beat the SQL with it.

6 Summary and concluding remarks

We have presented many aspects of QND measurements. Firstly we have introduced three different types of measurement and their errors. Then we did a brief recap of conventional quantum-mechanical measurement and shown that back-action originates from Heisenberg uncertainty relations. We have then put this recap on a more rigorous footing by introducing a framework for repeated position measurements. The introduced formalism can be used for many general systems and we have used it on a free particle for demonstration purposes. From that we have learned that QBA causes observables to undergo a random walk (which translates into random noise). Lastly, we have summarised a recent example of a QND measurement involving an optomechanical cavity. In this example, QBA was evaded, but the method did not yield better precision than the theoretically optimal precision of a non-QND measurement (the standard quantum limit). This was caused by technical limitations coming mainly from extraneous heating from optical absorption.

Presently, work is being put into figuring out how to overcome the technical limitations surrounding QND. For instance, an improved version of the experiment we cited with increased detection efficiency is being developed. Quantum non-demolition measurements are a promising way of improving the achievable detection efficiency and overcoming the standard quantum limit. This is important not only for practical reasons such as the enhancement of existing measurement devices, but it would also enable us to observe new quantum effects.

References

[1] T. J. Kippenberg and K. J. Vahala. Cavity optomechanics: Back-action at the mesoscale. *Science*, 321(5893):1172–1176, 2008.

[2] Carlton M. Caves, Kip S. Thorne, Ronald W. P. Drever, Vernon D. Sandberg, and Mark Zimmermann. On the measurement of a weak classical force coupled to a quantum-mechanical oscillator. i. issues of principle. *Rev. Mod. Phys.*, 52:341–392, Apr 1980.

[3] V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne. Quantum nondemolition measurements. *Science vol. 209 iss. 4456*, 209, aug 1980.

[4] Carlton M. Caves and G. J. Milburn. Quantum-mechanical model for continuous position measurements. *Phys. Rev. A*, 36:5543–5555, Dec 1987.

[5] Bonny L. Schumaker. Quantum mechanical pure states with gaussian wave functions. *Physics Reports vol. 135 iss. 6*, 135, apr 1986.

[6] I. Shomroni, L. Qiu, and D. Malz et al. Optical backaction-evading measurement of a mechanical oscillator. *Nat Commun*, 10, 2019.